Math Contest 1 Solutions

1. The answer is **3**. There are eight people at the party, but the most handshakes anyone can receive is 6. So, the 7 numbers in the set $\{0, 1, 2, 3, 4, 5, 6\}$ must be the number of handshakes that the 7 people other than the hostess have given. Consider the person who shakes 6 hands. He must shake hands with those who received 1-5 handshakes. Since this is only 5 people, he must also shake hands with the hostess. Notice that he must be the only one who shakes the hand of the person who receives 1 handshake. Consider the person shaking 5 hands. He cannot give a handshake with the person receiving 1 or 0 handshakes. So, he must shake hands with those who received 2-6 handshakes. So, he must shake the hand of the hostess too. Similarly, the person giving 4 handshakes must shake the hand of the hostess as well as those receiving 3-6 handshakes. The people receiving 3 or fewer handshakes are now accounted for, and no one else can shake the hostess' hand.

2. The answer is $\mathbf{\binom{7}{2} \cdot 9^5 + \binom{7}{3} \cdot 8 \cdot 9^4 = 3077109}$. It matters whether the eight digit number begins with 1 or not.

   - **Case 1:** If the first digit is a 1, there are seven slots left in which to put the remaining two 1's. So, there are $\binom{7}{2}$ ways to select the slots for the two 1's. The remaining 5 slots can be filled with any of 9 digits: $0, 2, 3, \ldots, 9$. So, if the first digit is 1, there are $\binom{7}{2} \cdot 9^5$ ways of creating 8-digit numbers with three 1's.
   - **Case 2:** If the first digit is not a 1, there are seven slots left in which to put the three 1's. So, there are $\binom{7}{3}$ ways to do this. The first slot has 8 possibilities, because it cannot hold either a 1 or a 0. The remaining 4 slots can be filled with any of the 9 digits which are not 1. So, there are $\binom{7}{3} \cdot 8 \cdot 9^4$ many ways.

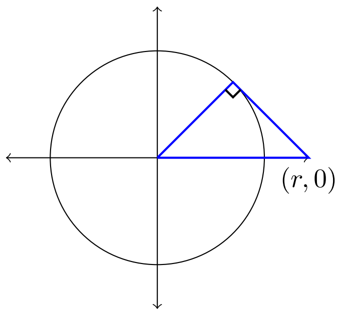
3. The answer is **8%** or **.08**. The total probability is 1 (or 100%). There are 50 numbers ≤ 50, i.e. the numbers 1 through 50. The probability of picking one of those is $p \cdot 50$. There are 50 numbers > 50, i.e. the numbers 51 through 100. The probability of picking one of those is $3p \cdot 50$. So, $50p + 3p50 = 1 \Rightarrow 200p = 1 \Rightarrow p = \frac{1}{200}$. Now, we just have to count the perfect squares. They are 1,4,9,16,25,36,49,64,81,100. 1 through 49 each have a $\frac{1}{200}$ chance of being picked, while 64, 81, and 100 each have a $\frac{3}{200}$ chance of being picked. So, the total probability is sum of these probabilities:

$$\frac{1+1+1+1+1+1+1+3+3+3}{200} = \frac{8}{100} = \mathbf{\frac{2}{25}} = \mathbf{.08}.$$

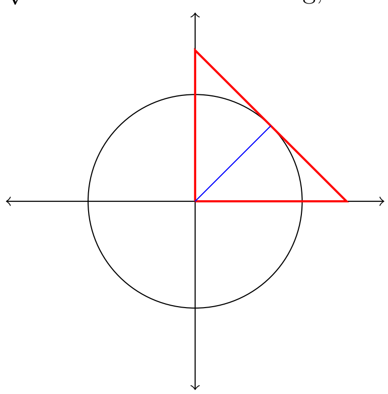
4. The answer is $\mathbf{r = 2}$. The key fact is that any line tangent to a circle forms a right angle with the radius of that circle. I will give two proofs: an algebraic proof and a geometric proof.

Algebraic: The line $y + x = r$ forms a right angle with the line through origin, represented by the radius. Since the slope of $y+x = r$ is -1, the slope of the perpendicular line is 1. Since the perpendicular line goes through $(0, 0)$, the equation of that line is $y = x$. So, at the point where $y + x = r$ touches the circle, $y = x$. This simplifies our two equations. At the point of tangency, it must be true that $x+x = r$ and $x^2+x^2 = r$. Solving the first equation for x gives $x = \frac{r}{2}$. Since the second equation simplifies to $2x^2 = r$, we substitute for x to get $2\left(\frac{r}{2}\right)^2 = r \Rightarrow \frac{r^2}{2} = r \Rightarrow r^2 - 2r = 0$. So, either $r = 0$ (which is silly) or $r = 2$.

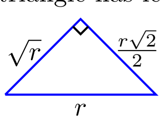
Geometric: Form the right triangle in blue below.



The hypotenuse is r, because that is the length along the x axis. The leg which is the radius is $\sqrt{r}$. To find the second leg, draw the segment of the line $y + x = r$ from the x to the y axis.



This is an isosceles right triangle with leg $= r$, so the hypotenuse is $r\sqrt{2}$. The radius forms two congruent triangles, so the radius bisects the hypotenuse. Thus, the missing side in the blue triangle has length $\frac{r\sqrt{2}}{2}$. Thus, we have the triangle below.



Now, use the Pythagorean Theorem.

$$(\sqrt{r})^2 + \left(\frac{r\sqrt{2}}{2}\right)^2 = r^2$$

$$r + \frac{2r^2}{4} = r^2$$

Combining and setting $= 0$, we have $\frac{r^2}{2} - r = 0$ which factors to be $r\left(\frac{r}{2} - 1\right) = 0$. So, either $r = 0$ (which is still silly) or $r = 2$.

5. The answer is $\mathbf{m = \pm 1}$. (Both numbers are needed for full credit.) We assume that $m + \frac{1}{m} = n$, where n is an integer. Multiplying both sides by m gives us $m^2 + 1 = nm$. Moving to all to the left side gives $m^2 - nm + 1 = 0$. Use the quadratic formula to solve. $m = \dfrac{n \pm \sqrt{n^2 - 4}}{2}$. Since m is rational, the numerator, $n \pm \sqrt{n^2 - 4}$ must be an integer. That means $n^2 - 4$ must be a perfect square. So, n^2 is a perfect square and $n^2 - 4$ is also a perfect square. But, perfect squares are always an odd number apart. The only possible explanation is that, $n^2 - 4 = 0$. So, $n = \pm 2$.

   Putting this all together, $m + \frac{1}{m} = 2$ or $m + \frac{1}{m} = -2$. Thus, $m = \pm 1$.

6. The answer is $\mathbf{\frac{7}{9}}$. The total area is 11. So, we are looking for a line that creates an area of $\frac{11}{2}$. If you draw a line from $(0,0)$ through $(3,1)$, this creates an area of $\frac{7}{2}$ in the bottom piece. So, draw a line as shown that intersects a distance x from the top. The top shape is a trapezoid, with area $\frac{1}{2}(3+x)3$. (The two bases are 3 and x in length respectively. The height is 3.) This area must be $\frac{11}{2}$. So, solving $\frac{1}{2}(3+x)3 = \frac{11}{2}$ for x, we have $3(3+x) = 11$ and thus $9 + 3x = 11$. So, $x = \frac{2}{3}$. So, the line must pass through the origin and the point $(3, 3 - \frac{2}{3})$, which is $(3, \frac{7}{3})$. And the slope is

$$\frac{\frac{7}{3} - 0}{3 - 0} = \frac{\frac{7}{3}}{3} = \mathbf{\frac{7}{9}}$$